# MAT 157 Analysis I

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# Contents

## 0.1 Introduction

The course is mainly based on the book *Spivak Calculus* covers chapter 10 of *A readable introduction to real mathematics* by *Daniel Rosenthal et al.*

# Chapter 1

# Week 1

## 1.1 Cardinality

**Definition 1.1.1: Cardinality**

Cardinality is the "number of things" in a set. Example: $A = \{1, 2, 3\}$ has cardinality 3, $B = \{1, 2, 3, ...\}$ has infinite cardinality. To be specific, when counting the numbers in the set, the mapping of numbers to the elements need to be **bijective**.

**Definition 1.1.2**

Given two sets $S$ and $T$, if there is a $f : S \longrightarrow T$, that's bijective,$S$ and $T$ have the same cardinality, we write $|S| = |T|$.

**Definition 1.1.3**

Given a bijective function $f : S \longrightarrow T$ then $S$ and $T$ have the same cardinality

**Note:-**

Note that the concepts of injective, surjective and bijective functions are noted down in course note MAT 240, thus disregard

**Example 1.1.1** (Bijection from $[1, 3] \longrightarrow (30, 40]$)

(This is a thinking question at the end of the week 1 lecture) Construct a bijection between [1,3] to (30,40]
A solution could be that

$$f(x) = \begin{cases} 30 + \frac{1}{2}, & x = 1, \\ 30 + \frac{1}{2^{n+1}}, & x = 1 + \frac{2^{-n}}{5}, n \in \mathbb{N} \\ 5x + 25, & \text{otherwise.} \end{cases}$$

The basic idea is that the sequence $\frac{1}{2^n}, n \in \mathbb{N}$ is strictly decreasing from $\frac{1}{2}$ approaching 0. This allows us to construct a sequence $30 + \frac{1}{2^n}$ which is decreasing from $30 + \frac{1}{2}$ to 30. By shifting it to the left, we have $30 + \frac{1}{2^{n+1}}$, which gives a vacant spot for assigning the case where $x = 1$ (the idea of Hilbert's hotel).
The second case's condition may look strange. The since we have pushed te $30 + \frac{1}{2^n}$ sequence back by one, we need to remap the preimage of it to the new one. The original function is $f(x) = 5x + 25$, so the condition derives from solving the equation $30 + \frac{1}{2^n} = 5x + 25$, which gives $x = 1 + \frac{2^{-n}}{5}$
In short, the key idea is to find a decreasing sequence in the codomain that's approaching the open end, and shift it to allow a space for $x = 1$.

**Definition 1.1.4**

A set $A$ is finite if there is a $n \in \mathbb{N}$ and a $f : \{1, 2, ..., n\} \to A$ which is bijective. We say $|A| = n$

**Definition 1.1.5**

If $A$ is the empty set then $|A| = |\emptyset| = |\{\}| = 0$

**Definition 1.1.6**

A set $A$ is **countable** if $A$ is infinite or $|A| = |\mathbb{N}|$. There is $f : \mathbb{N} \to A$ is bijective

**Example 1.1.2** (Even natural numbers have same cardinality as all natural numbers )

**Question 1**

Let $\mathbb{E}$ = even natural numbers = $\{2, 4, 6, 8, ...\}$. $|\mathbb{E}| = |\mathbb{N}|$ and $E$ is countable. Find a function $f : \mathbb{N} \to \mathbb{E}$, where $f$ is bijective.

***Solution:***

$$f(n) = 2n$$

**Claim 1.1.1** $f(n) = 2n$ is injective

***Proof:*** Assume $f(n_1) = f(n_2)$ where $n_1, n_2 \in \mathbb{N}$. Then $2n_1 = 2n_2$ and so $2(n_1 - n_2) = 0$. Therefore $n_1 - n_2 = 0$ and so $n_1 = n_2$. Thus $f$ is injective. ☺

**Claim 1.1.2** $f(n) = 2n$ is surjective

***Proof:*** Let $t \in \mathbb{E}$. Then $t = 2n$ for some $n \in \mathbb{N}$. Therefore $f(n) = t$. And $f$ is onto $\mathbb{E}$. ☺

**Example 1.1.3** (The set of even natural numbers and the set of odd natural numbers have the same cardinality.)

***Proof:*** Let the set of even natural numbers $E = \{2, 4, ..., 2n, ..\}$ and the set of odd natural numbers $O = \{1, 3, ..., 2n - 1, ...\}$ To satisfy the definition 1.1, we need to show that there is a injective function taking $E$ onto $O$. Define a function $f : E \to O$, simply note that $k_1 - 1 = k_2 - 1$ implies $k_1 = k_2$. Also, $f$ is surjective, because for any $m \in O$, $m = 2n - 1$ for some $n \in \mathbb{N}$. Therefore $f(2n) = 2n - 1 = m$. Thus, $f$ is bijective and $|E| = |O|$. ☺

**Example 1.1.4** (The set of natural numbers and the set of nonnegative integers have the same cardinality.)

***Proof:*** Let $S$ denote the set of nonnegative integers. We want to construct a bijective function $f : S \to \mathbb{N}$. Let $f$ be $f(n) = n + 1, n \in S$. This makes $f$ surjective. Also, $f(n_2) = f(n_2)$ implies $n_1 + 1 = n_2 + 1$, giving $n_1 = n_2$. Since $f$ is bijective, $\mathbb{N}$ and $S$ have the same cardinality. ☺

**Theorem 1.1.1** The set of natural numbers and the set of positive rational numbers have the same cardinality

***Proof:*** To prove the theorem, we first construct the array:

$$\begin{array}{cccccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \frac{2}{7} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \frac{3}{7} & \cdots \\ \frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \frac{4}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Let the following:

- $f(1) = \frac{1}{1}$
- $f(2) = \frac{1}{2}$
- $f(3) = \frac{2}{1}$
- $f(4) = \frac{3}{1}$
- $f(5) = \frac{1}{3}$
- $f(6) = \frac{1}{4}$
- $f(7) = \frac{2}{3}$
- $f(8) = \frac{3}{2}$
- $f(9) = \frac{4}{1}$
- $f(10) = \frac{5}{1}$
- ...

We can observe that for the first diagonal of equivalent elements $1/1, 2/2, ...$ it moves 1 down 1 right, $1d1r$ in short. Next for $1/2$ it moves $1d2r$ and $1d3r$ for $1/3$, and so on. Moreover, for $2/1, 4/2, ...$ it move $2d1r$, and $3/1, 6/2, ...$ moves $3d1r$, and so on. According to this sequence, we can pair natural numbers to the positive rational numbers indicated in the "zigzagging" manner. Therefore $|\mathbb{Q}^+| = |\mathbb{N}|$
To be precise, consider the set

$$S = \{(m, n) \in \mathbb{N} \times \mathbb{N} : gcd(m, n) = 1\}$$

Next define the function $f$:

$$f : \mathbb{N} \to \mathbb{Q} > 0, f(x) = \frac{m_x}{n_x}$$

.

**Claim 1.1.3** $f$ is injective

***Proof:*** If $f(x) = f(y)$,

$$\frac{m_x}{n_x} = \frac{m_y}{n_y}$$

since both fraction have gcd = 1,$x = y$ 😄

**Claim 1.1.4** $f$ is surjective

***Proof:*** Take any $q \in \mathbb{Q}_+$, it can be expressed as $q = \frac{m}{n}$ in lowest terms. Then $(m, n) \in S$, which equals to some $a_x, b_x$), thus $f(x) = q$ 😄

Since the function $f$ is bijective, $|\mathbb{N}| = |\mathbb{Q}_{>0}|$

😄

Another proof (during lecture)

***Proof:*** Define

$$R = \{\frac{q}{p} : p, q \in \mathbb{N}\}$$

Clearly $\mathbb{Q}^+ \subsetneq R$, since, for example $\frac{2}{4} \in R$ but when considered as a set of rationals we have $2/4 = 1/2 \in \mathbb{Q}^+$.

**Lemma 1.1.1**

Any subset of a countable set is either finite or countably infinite.

By proving $R$ is countable, it follows that $\mathbb{Q}^+$ is countable as well. Consider the map

$$\psi : \mathbb{N} \times \mathbb{N} \to R$$
$$\psi(p, q) = \frac{q}{p}$$

This is a **surjection**. Now we need to prove that $\mathbb{N} \times \mathbb{N}$ is countable. We define the **Cantor pairing function** $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,

$$g(i, j) = \frac{(i + j - 2)(i + j - 1)}{2} + j$$

This enumerates all lattice points $(i, j)$ by diagonals of constant sum $i + j$. By checking it's bijective (skip), $\mathbb{N} \times \mathbb{N}$ is countable. Taking the inverse $f = g^{-1} : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Since $\psi$ is surjective mapping $\mathbb{N} \times \mathbb{N}$ to $R$, $\varphi = \psi \circ f : \mathbb{N} \to R$ is a surjection. Therefore $R$ is countable. Applying the lemma, it implies that $\mathbb{Q}^+$ is countable. 😊

**Definition 1.1.7: Pigeonhole principle**

If $S$ is a **finite set**, then a function $g : S \to S$ is injective, if and only if it is onto. This principle fails for **infinite sets**.

***Proof:*** 😊

## 1.2 Countable Sets and Uncountable Sets

**Definition 1.2.1: Countable**

A set is **countable** (or denumerable or enumerable) if it is either

- **finite**
- or has the **same cardinality** as the set of natural numbers.

**Definition 1.2.2: Closed interval**

For $a$ and $b$ real numbers with $a \leqslant b$, the *closed interval from $a$ to $b$* is the set of all real numbers between $a$ and $b$, **including them**. It's denoted $[a, b] = \{x : a \leqslant x \leqslant b\}$

**Theorem 1.2.1**

The closed interval $[0, 1]$ is uncountable.

***Proof:*** We will show that there is no function $f$ maps $\mathbb{N}$ onto $[0, 1]$. The following is *Cantor diagonal argument*. Suppose that $f$ is any function taking $\mathbb{N} \to [0, 1]$. To prove that $f$ cannot be onto, we can imagine the following construction where we write out all the values of $f$ in a list, as follows:

$$\begin{aligned} f(1) &= .a_{11}a_{12}a_{13}a_{14}a_{15}\dots \\ f(2) &= .a_{21}a_{22}a_{23}a_{24}a_{25}\dots \\ f(3) &= .a_{31}a_{32}a_{33}a_{34}a_{35}\dots \\ f(4) &= .a_{41}a_{42}a_{43}a_{44}a_{45}\dots \\ f(5) &= .a_{51}a_{52}a_{53}a_{54}a_{55}\dots \\ &\vdots \end{aligned}$$

In other words, for $f(x) = 0.a_1a_2...$, where

$$a_k = \lfloor 10^k x \rfloor - 10\lfloor 10^{k+1}x \rfloor$$

Now we need to construct a number in $[0,1]$ but not in the range of function $f$. We consider the following sequence $x = .b_1b_2....$ where if $a_{ii} = 3, b_i = 4$, and if $a_{ii} \neq 3, b_i = 3$. In this way, we can construct a number $x$ that differs $f(j)$ in it's $j^{\text{th}}$ digit. Thus $f(j) \neq x \forall j$, so $x$ is not in the range of $f$. Moreover, if we try to adding the number $x$ into the range of $f$ by having a new function $g$ where $g(1) = x$ and $g(n) = f(n-1)$ for $n \geqslant 2$ (basically shifting every element in $f$ back by 1). However, the construction above could be used again on $g(x)$ and construct a new $x$ that is not in the range of $g$. Thus this proves that $[0,1]$ is not countable. 😄

To be more precise,

***Proof:*** Every $y \in [0,1]$ has binary expansion

$$y = \sum_{k \geqslant 1} a_k 2^{-k}, a_k \in 0, 1$$

Some $y$ has two expansions, e.g. $0.10000..._2$ and $0.0111..._2$.. For each $n \in \mathbb{N}$, a **convention** is that to write $f(n)$ in its binary expansion that's not eventually all 1's.
We can write

$$f(n) = 0.a_{n1}a_{n2}a_{n3}..., \ a_{nk} \in \{0, 1\}$$

Define a binary sequence where $b_k \geqslant 1$ by $b_k := 1 - a_{kk}$, let

$$x := 0.b_1b_2b_3... = \sum_{k \geqslant 1} b_k 2^{-k}$$

Then $x \in [0,1]$. This makes $x$ and $f(n)$ differ in the $n$-th digit, so $x \neq f(n)$. Since this holds for every $n$, we conclude $x \notin f(\mathbb{N})$. Therefore, no function $f : \mathbb{N} \to [0,1]$ is surjective. Hence $[0,1]$ is uncountable. 😄

**Theorem 1.2.2**
If $a$ and $b$ are real numbers and $a < b$, the $[a,b]$ and $[0,1]$ have the **same cardinality**.

***Proof:*** The theorem will be established if there is a bijective function $f : [0,1] \to [a,b]$. Let $f(x) = a + (b-a)x$. Then $f(0) = a$ and $f(1) = b$.
To prove it's **injective**, we have

$$\begin{aligned} f(x_1) &= f(x_2) \\ a + (b-a)x_1 &= a + (b-a)x_2 \\ (b-a)x_1 &= (b-a)x_2 \\ x_1 &= x_2 \end{aligned}$$

To show that $f$ is **surjective**, let $y$ be any element of $[a,b]$. Let $x = \frac{y-a}{b-a}$. Then $x \in [0,1]$ and $f(x) = y$. Thus, the $f$ is bijective, and $|[0,1]| = |[a,b]|$ 😄

**Theorem 1.2.3** The intervals $[0, 1]$ and $(0, 1]$ have the same cardinality

Similar proof to the thinking question 1.1, thus disregard.

**Theorem 1.2.4** If $|S| = |T|$ and $|T| = |U|$, then $|S| = |U|$

***Proof:*** Let $f : S \to T$ and $g : T \to U$ be **bijective**, and $h = g \circ f$. Given $u \in U$, since $g$ is surjective, there exists a $t \in T$ such that $g(t) = u$.
**Step 1: proving surjectivity**. Since $f$ is surjective, there s an $s \in S$ such that $f(s) = t$. Then $h(s) = g(f(s)) = g(t) = u$. Thus, $h$ is surjective.
**Step 2: proving injectivity**. Suppose that $h(s_1) = h(s_2)$, $g(f(s_1)) = g(f(s_2))$, so $f(s_1) = f(s_2)$ since $g$ is injective. Since $f$ is also injective, so $s_1 = s_2$. This proves that $h$ is bijective, and $|S| = |U|$. 😊

**Theorem 1.2.5** If $a, b, c, d$ are real numbers with $a < b$ and $c < d$, then $(a, b]$ and $(c, d]$ have the same cardinality

***Proof:*** $f$ is defined by $f(x) = a + (b - a)x$ which is a injective function mapping $(0, 1]$ onto $(a, b]$, as mentioned in the theorem 1.2. Hence, $|(0, 1]| = |(a, b]|$. Similarly $g$ is defined by $g(x) = c + (d - c)x$ which is a injective function mapping $(0, 1]$ onto $(c, d]$. Following theorem 1.2, $|(a, b]| = |(c, d]|$ 😊

**Theorem 1.2.6** The cardinality of the set of nonnegative real numbers is **the same as** the cardinality of the unit interval $[0, 1]$

***Proof:*** We aim to show the set $S = \{x : x \geqslant 1\}$ has same cardinality as $(0, 1]$.
**Step 1.1: show surjectivity**. Note that the function $f$ is defined by $f(x) = \frac{1}{x}$ maps $S$ **into** $(0, 1]$; for if $x \geqslant 1$, then $\frac{1}{x} \leqslant 1$. Also, $f$ maps $S$ onto $(0, 1]$; for if $y \in (0, 1]$,then $\frac{1}{y} \geqslant 1$ and $f(\frac{1}{y}) = y$.
**Step 1.2: show injectivity**. To see that $f$ is injective, suppose that $f(x_1) = f(x_2)$, then $\frac{1}{x_1} = \frac{1}{x_2}$, so $x_1 = x_2$. Hence, $f$ is bijective. Thus, $|S| = |(0, 1]$.
**Step 2** Now let $T = \{x : x \geqslant 0\}$. Define the function $g$ by $g(x) = x - 1$. Then $g$ is a bijective function mapping $S \to T$. Hence, $|T| = |S|$. Thus, by theorem 1.2, $|T| = |(0, 1]$. By theorem 1.2,

$$|[0, 1]| = |(0, 1]| \Rightarrow |T| = |[0, 1]|$$

😊

**Theorem 1.2.7** The union of a countably many countable sets is countable

That is, if $\{S_i : i \in \mathbb{N}\}$ is a countable collection of sets and each $S_i$ is countable, then

$$S = \bigcup_{i=1}^{\inf} S_i$$

is countable.

The proof varies from the book

***Proof:*** For each $i \in \mathbb{N}$, since $S_i$ is countable, there exists a **surjection**

$$f_i : \mathbb{N} \to S_i$$

Now defining a function

$$F : \mathbb{N} \times \mathbb{N} \to S$$
$$F(i, j) = f_i(j)$$

This is a **surjection** because for any $s \in S$, exists $s \in S_k$ for some $k$, so $s = f_k(j)$ for some $j$. Hence $s = F(k, j)$.
**Step 1: Countability of $\mathbb{N}\times\mathbb{N}$**. Previously proven with **Cantor's pairing function** that it's countable. We can define

$$g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$g(i, j) = \frac{(i + j - 2)(i + j - 1)}{2} + j$$

this function is a bijection, hence $\mathbb{N} \times \mathbb{N}$ is countable.
**Step 2: Composition**. Since $g$ is bijective, it has an inverse $f = g^{-1} : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, by compositing it with $F$ gives:

$$h = F \circ f : \mathbb{N} \to S$$

Then $h$ is a surjection from $\mathbb{N} \to S$. By definition, a set is countable is there is a surjection from $\mathbb{N}$ onto it. Thus, the function $h$ constructed above means that $S$ is countable. 😄

**Theorem 1.2.8** Cantor-Bernstein Theorem

If there are injections $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \to B$.

# Chapter 2

# Week 2

**Note:-**

This is the end of contents in *Rosenthal's* book. The following contents are mainly based on *Spivak's* Calculus

**Definition 2.0.1: Basic properties of numbers**

The properties are listed in notes for Rudin's analysis, rewritten in the format of

1. **Associativity** For any $a, b, c \in A$ , $a + (b + c) = (a + b) + c$
2. **Additive identity** There is $0 \in A$ so that for any $a \in A, 0 + a = a + 0 = a$
3. **Additive inverse** For any $a \in A$, there is $-a \in A$ so that $a + (-a) = (-a) + a = 0$
4. **Commutativity** For any $a, b \in A, a + b = b + a$
5. **Associativity of multiplication** For any $a, b, c \in A$, $a(bc) = (ab)c$
6. **Multiplicative identity** There is $1 \in A, 1 \neq 0$ so that for any $a \in A, 1a = a1 = a$
7. **Multiplicative inverse** For any $a \in A, a \neq 0$, there is $a^{-1} \in A$ so that $aa^{-1} = a^{-1}a = 1$
8. **Multiplicative commutativity** For any $a, b \in A, ab = ba$
9. **Distributive law** For any $a, b, c \in A, a(b + c) = (ab) + (ac)$
10. **Trichotomy law** Consider the collection of all positive numbers, $P$. For every number $a$, one and only one of the following holds:
    (a) $a = 0$
    (b) $a$ is in the collection $P$
    (c) $-a$ is in the collection $P$
11. **Closure under addition** If $a$ and $b$ are in $P$, then so is $a + b$
12. **Closure under multiplication** If $a$ and $b$ are in $P$, then so is $a \cdot b$

$P$ is the collection of all positive numbers.

**Claim 2.0.1** If $(A, +)$ satisfies $(1) - (3)$, then $0$ is unique

***Proof:*** Suppose $\tilde{0} \in A$ also satisfies the identity property, i.e.

$$a + \tilde{0} = a = \tilde{0} + a \quad \text{for all } a \in A.$$

Let $a \in A$ be arbitrary. Then

$$a + \tilde{0} = a.$$

By property (3), the inverse $-a$ exists. Adding $-a$ to both sides gives

$$(-a) + (a + \tilde{0}) = (-a) + a.$$

By associativity (1),

$$((-a) + a) + \tilde{0} = (-a) + a.$$

By property (3), $(-a) + a = 0$. Hence

$$0 + \tilde{0} = 0.$$

Finally, by the identity property (2),

$$\tilde{0} = 0.$$

Thus the additive identity is unique. 😊

**Question 1**

- uniqueness of $-a \Rightarrow -(-a) = a$
- prove $0 \cdot a = 0$ for all $a \in A$
- prove 1 is unique
- $(-a) \cdot b = -(a \cdot b)$
- $(-a) \cdot (-b) = a \cdot b$

Spivak's numbers means a set that contains $\mathbb{Z}$ and satisfies (1)-(9).

**Definition 2.0.2: Ordering or $A$**

If $a, b \in A$ and $A, P$ satisfies (10)-(12) then,

- $a > b$ means $a + (-b) \in P$
- $a \geqslant b$ means $a + (-b) \in P$ or $a + (-b) = 0$
- $a < b$ means $b + (-a) \in P$
- $a \leqslant b$ means $b + (-a) \in P$ or $b + (-a) = 0$

In particular, $a > 0$ if and only if $a \in P$.

**Claim 2.0.2**

**Lemma 2.0.1** If $a \in P \Rightarrow a + 0 \in P$

- If $a < b$ and $c > 0$ then $a \cdot c < b \cdot c$
- If $a < b$ and $c \in A$ then $a + c < b + c$
- If $a < b$ and $c < 0 \Rightarrow a \cdot c > b \cdot c$

***Proof:*** Proof of (2), from definition of $<$, $a < b \Rightarrow b + (-a) \in P$.

$$\begin{aligned}\Rightarrow b + (-a) + 0 \in P \\ b + (-a) + ((-c) + c) \in P \\ (b + (-c)) + ((-a) + c) \in P \\ (b + (-c)) + (-(a + (-c))) \in P\end{aligned}$$

😄

**Claim 2.0.3** If $0 < a < b \Rightarrow a^2 < b^2$

***Proof:*** Proof 1:

$$b^2 + (-a^2) = (b \cdot b) + (-(a \cdot a)) = (b + (-a))(b + a)$$

Since $b > a$, $b - a > 0$ and $b + a > 0$. This proves $(b - a)(b + a) > 0 \Rightarrow (b^2 - a^2) > 0 \Rightarrow b^2 > a^2$
Proof 2:

$$\begin{aligned}a^2 &< a \cdot b \\ a \cdot b &< b^2 \\ a^2 &< b^2\end{aligned}$$

😄

**Claim 2.0.4** If $a > 0, b > 0, a^2 < b^2 \Rightarrow a < b$

**Claim 2.0.5** If $a < b$ then $a^3 < b^3$

**Claim 2.0.6** If $0 < a < b$, $a, b \in \mathbb{R}$, then $a < \sqrt{a \cdot b} < \frac{a-b}{2} < b$
Assume that for any $c > 0, c \in \mathbb{R}$ there is a unique $d \in \mathbb{R}, d > 0$ so that $d^2 = c$, call $d$ "$\sqrt{c}$".

# Chapter 3

# Week 3

## 3.1 Peano Axiom

The Peano's axioms are proposed by **Giuseppe Peano (1889)** to rigorously define the natural numbers $\mathbb{N} = \{0, 1, 2, 3, ...\}$ and their arithmetic.

**Definition 3.1.1: Natural numbers**

The natural numbers $\mathbb{N}$ are a set equipped with a successor operation $S : \mathbb{N} \to \mathbb{N}$ such that the following Peano axioms hold:

1. **Existence of the first element:** 1 is a natural number.
2. **Successor property:** If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$.
3. **Injectivity of successor:** If $m, n \in \mathbb{N}$ and $S(m) = S(n)$, then $m = n$.
4. **Non-predecessorship of 1:** There is no $n \in \mathbb{N}$ such that $S(n) = 1$.
5. **Induction axiom:** If $A \subseteq \mathbb{N}$ such that $1 \in A$, and whenever $n \in A$ implies $S(n) \in A$, then $A = \mathbb{N}$.

Using these axioms, we can define addition and multiplication recursively via $S$.

**Note:-**

Peano originally had $\mathbb{N}$ start with 1, later $\mathbb{N}$ started with 0. We take $\mathbb{N}$ start with 1 for the course.

## 3.2 Induction

Here are some examples for mathematical induction during lecture:

**Example 3.2.1** (For all $n \in \mathbb{N}$: $2 + 4 + 6 + ... + 2n = n(n + 1)$)

The ideas:

- Let $K = \{n \in \mathbb{N}, 2 + 4 + ... + 2n = n(n + 1)\}$
- show $1 \in K$
- assume $n \in K$ show $n + 1 \in K$
- Conclude $K = \mathbb{N}$ by peano axiom that $S(n) = n + 1$

***Proof:*** Let $K = \{n \in \mathbb{N}, 2 + 4 + ... + 2n = \sum_{i=1}^{n} 2i = n(n + 1)\}$. Take $n = 1$, then $2 = 1(1 + 1)$ is true. So $1 \in K$.
Assume $n \in K$,

$$2 + 4 + 6 + ... + 2n + 2(n + 1) = n(n + 1) + 2(n + 1)$$

because $n \in K$
So

$$2 + 4 + .... + 2(n+1) = (n+1)((n+1)+1)$$

is true, and so $n+1 \in K$. Therefore $K$ satisfies Peano axiom 5, and so $K = \mathbb{N}$.

😊

**Example 3.2.2**

Let $a \in \mathbb{R}$ with $-k < a < 0$ and $a < -1$ For all $n \in \mathbb{N}$

$$(1+a)^n < 1 + na + \frac{n^2a^2}{2}$$

***Proof:*** Let

$$K = \{n \in \mathbb{N} : (1+a)^n < 1 + na + \frac{n^2a^2}{2}\}$$

We first show $1 \in K$,

$$(1+a) < 1 + 1 \cdot a + \frac{1^2a^2}{2}$$

because $\frac{1^2a^2}{2} > 0$ and $a \neq 0$. Therefore $1 \in K$.
Assume $n \in K$, so

$$(1+a)^n < 1 + na + \frac{n^2a^2}{2}$$

We want to show that $n+1 \in K$.

$$(1+a)^{n+1} = (1+a)^n(1+a) < (1 + na + \frac{n^2a^2}{2})(1+a)$$

because $n \in K$ and $1 + a > 0$. By expanding product,

$$\begin{aligned}
(1+a)^{n+1} &< 1 + (n+1)a + na^2 + \frac{n^2a^2}{2} + \frac{n^2a^3}{2} \\
&= 1 + (n+1)a + \frac{(n+1)^2a^2}{2} - \frac{(n+1)^2a^2}{2} + na^2 + \frac{n^2a^2}{2} + \frac{n^2a^2}{3} \\
\text{by several steps of deduction,} &= 1 + (n+1)a + \frac{(n+1)^2a^2}{2} + \frac{a^2(an^2-1)}{2} \\
&\leqslant 1 + (n+1)a + \frac{(n+1)^2a^2}{2}
\end{aligned}$$

because $a < 0 \Rightarrow \frac{a^2(an^2-1)}{2} < 0$. Therefore $n+1 \in K$ and $K = \mathbb{N}$ 😊

**Theorem 3.2.1** (AM–GM Inequality)

Let $x_1, x_2, \ldots, x_n \geq 0$. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

***Proof:*** We proceed by induction on $n$.
**Base case:** $n = 1$. Trivially $\frac{x_1}{1} = x_1 = \sqrt[1]{x_1}$.
**Base case:** $n = 2$. We need to show

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}.$$

Consider the nonnegative square

$$0 \leq (x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2.$$

Rearrange:

$$x_1^2 + 2x_1x_2 + x_2^2 \geq 4x_1x_2 \implies (x_1 + x_2)^2 \geq 4x_1x_2.$$

Taking the positive square root (since both sides are nonnegative) and then dividing by 2 gives the desired inequality. Equality holds only when $(x_1 - x_2)^2 = 0$, i.e. $x_1 = x_2$.
**Induction step.** Assume the result holds for all lists of length $n$. Let $x_1, x_2, \ldots, x_{n+1} \geq 0$. Write

$$\alpha = \frac{x_1 + \cdots + x_{n+1}}{n+1}.$$

If all $x_i = \alpha$, then both sides (AM and GM) equal $\alpha$, and the result holds with equality. Otherwise, not all $x_i$ are equal.
We can reorder so that $x_{n+1}$ is one of the "extreme" elements, and consider the first $n$ elements $x_1, \ldots, x_n$. By the induction hypothesis,

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Now compare the arithmetic mean of all $n+1$ with the geometric mean:

$$\alpha = \frac{1}{n+1}(x_1 + \cdots + x_n + x_{n+1}) = \frac{n}{n+1} \cdot \frac{x_1 + \cdots + x_n}{n} + \frac{1}{n+1}\, x_{n+1}.$$

Applying the (already established) two–term AM–GM to the two numbers

$$\frac{x_1 + \cdots + x_n}{n} \quad \text{and} \quad x_{n+1},$$

we get

$$\frac{1}{2}\left(\frac{x_1 + \cdots + x_n}{n} + x_{n+1}\right) \geq \sqrt{\frac{x_1 + \cdots + x_n}{n} \cdot x_{n+1}}.$$

One arranges the coefficients so that this gives

$$\alpha \geq \sqrt[n+1]{x_1 x_2 \cdots x_{n+1}}.$$

A more detailed algebraic manipulation is usually given in textbooks (or see the Wikipedia "induction proof" section). Equality can hold only if each inequality used in the induction is an equality, which forces $x_1 = \cdots = x_{n+1}$.

😄

**Theorem 3.2.2** (AM–GM an alternative proof)
For any $n \in \mathbb{N}$ and any positive real numbers $b_1, \ldots, b_n$,

$$\frac{b_1 + \cdots + b_n}{n} \;\geq\; (b_1 b_2 \cdots b_n)^{1/n},$$

with equality iff $b_1 = \cdots = b_n$.

***Proof:*** We split the proof in two steps.

**Step 1 (Inductive lemma for product = 1).** *Claim.* If $a_1, \ldots, a_n > 0$ and $a_1 \cdots a_n = 1$, then

$$a_1 + \cdots + a_n \;\geq\; n,$$

with equality iff $a_1 = \cdots = a_n = 1$.
*Proof by induction on $n$.*

- **Base $n = 1$.** From $a_1 \cdot 1 = 1$ we get $a_1 = 1$, hence $a_1 \geq 1$ with equality.
- **Inductive step.** Assume the statement holds for some $n \geq 1$. Let $a_1, \ldots, a_{n+1} > 0$ satisfy $a_1 \cdots a_{n+1} = 1$. Reindex so that

$$a_n = \min\{a_1, \ldots, a_{n+1}\} \quad \text{and} \quad a_{n+1} = \max\{a_1, \ldots, a_{n+1}\}.$$

Because the product is 1, we must have $a_n \le 1 \le a_{n+1}$. Hence

$$(a_n - 1)(a_{n+1} - 1) \le 0 \quad \Longrightarrow \quad a_n + a_{n+1} \ \ge \ 1 + a_n a_{n+1}. \tag{$*$}$$

Now define the $n$–tuple

$$b_1, \ldots, b_{n-1}, b_n \ := \ a_1, \ldots, a_{n-1}, \ a_n a_{n+1}.$$

Then $b_1 \cdots b_n = (a_1 \cdots a_{n-1})(a_n a_{n+1}) = a_1 \cdots a_{n+1} = 1$. By the inductive hypothesis,

$$a_1 + \cdots + a_{n-1} + a_n a_{n+1} \ \ge \ n. \tag{$\dagger$}$$

Adding ($*$) and ($\dagger$) gives

$$a_1 + \cdots + a_{n+1} \ = \ \big(a_1 + \cdots + a_{n-1} + a_n a_{n+1}\big) + \big(a_n + a_{n+1} - a_n a_{n+1}\big) \ \ge \ n + 1.$$

*Equality case.* In ($*$) equality forces $a_n = 1$ and $a_{n+1} = 1$; then ($\dagger$) forces $a_1 = \cdots = a_{n-1} = 1$ by the inductive hypothesis. Thus equality holds iff all $a_i = 1$.

This completes the induction and proves the claim. △

**Step 2 (Normalization and conclusion).** Let $b_1, \ldots, b_n > 0$ and set

$$G := (b_1 b_2 \cdots b_n)^{1/n}, \qquad a_i := \frac{b_i}{G} \quad (1 \le i \le n).$$

Then $a_1 \cdots a_n = \frac{b_1 \cdots b_n}{G^n} = 1$, so the lemma yields

$$\sum_{i=1}^{n} a_i \ = \ \sum_{i=1}^{n} \frac{b_i}{G} \ \ge \ n.$$

Multiplying by $G$ gives

$$b_1 + \cdots + b_n \ \ge \ n\,G \ = \ n\,(b_1 b_2 \cdots b_n)^{1/n},$$

and dividing by $n$ proves AM–GM.
*Equality case.* Equality in Step 1 requires $a_1 = \cdots = a_n = 1$, i.e. $b_1 = \cdots = b_n = G$. 😄

**Definition 3.2.1: Comparison: Standard vs. Strong Induction**

**Standard (Weak) Induction.** To prove a statement $P(n)$ for all $n \geqslant n_0$:

- **Base case:** Show $P(n_0)$ is true.
- **Inductive step:** Assume $P(k)$ is true for some $k \geqslant n_0$ (the induction hypothesis). Prove $P(k+1)$ is true.

Here the hypothesis only assumes $P(k)$ to prove $P(k+1)$.

**Strong Induction.** To prove a statement $P(n)$ for all $n \geqslant n_0$:

- **Base case:** Show $P(n_0)$ is true (and sometimes $P(n_0+1), \dots, P(n_0+r)$ depending on the problem).
- **Inductive step:** Assume $P(n_0), P(n_0+1), \dots, P(k)$ are all true (the strong induction hypothesis). Prove $P(k+1)$ is true.

Here the hypothesis allows you to assume *all* earlier cases up to $k$, not just the immediate predecessor.

**Summary.**

- Standard induction: assume $P(k)$ to prove $P(k+1)$.
- Strong induction: assume all $P(j)$ for $n_0 \leqslant j \leqslant k$ to prove $P(k+1)$.
- Both principles are logically equivalent, but strong induction is often easier when $P(k+1)$ depends on several earlier cases.

**Definition 3.2.2: Fundamental Theorem of Arithmetic**

If $n \in \mathbb{N}, n \geqslant 2$ can be written as a product of primes.

***Proof:*** We proceed by *strong induction* on $n$.
**Base case ($n = 2$).** 2 is prime, so it is already a product of primes (just itself).
**Induction hypothesis.** Assume that every integer $m$ with $2 \leqslant m \leqslant k$ can be written as a product of primes.
**Inductive step.** We prove the statement for $k+1$.

- If $k+1$ is prime, then it is its own prime factorization.
- If $k+1$ is composite, then there exist integers $a, b$ with $2 \leqslant a, b \leqslant k$ such that $k+1 = ab$. By the induction hypothesis, both $a$ and $b$ can be written as products of primes. Multiplying these factorizations gives a prime factorization of $k+1$.

**Conclusion.** By the principle of strong induction, every $n \geqslant 2$ has a prime factorization. ☺

### 3.2.1 Recursion Relations

**Example 3.2.3** (Recursion Relations)

Generate a sequence $\{c_1, c_2, \dots\} \subseteq \mathbb{N}$ as follows:

$$c_1 = 0, \quad c_2 = 1, \quad c_{n+1} = 5c_n - 6c_{n-1} \quad (n \geqslant 2).$$

**Claim 3.2.1**

For all $n \geqslant 1$, the closed form is

$$c_n = 3^{n-1} - 2^{n-1}.$$

***Proof:*** We proceed by induction on $n$.

**Base cases.** For $n = 1$: $c_1 = 0$, and the formula gives $3^0 - 2^0 = 1 - 1 = 0$. For $n = 2$: $c_2 = 1$, and the formula gives $3^1 - 2^1 = 3 - 2 = 1$. So the formula holds for $n = 1, 2$.
**Induction hypothesis.** Assume for some $n \geqslant 2$ that

$$c_n = 3^{n-1} - 2^{n-1}, \qquad c_{n-1} = 3^{n-2} - 2^{n-2}.$$

**Inductive step.** Using the recurrence:

$$c_{n+1} = 5c_n - 6c_{n-1}.$$

Substitute the hypothesis:

$$c_{n+1} = 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2}).$$

Simplify:

$$= 5 \cdot 3^{n-1} - 5 \cdot 2^{n-1} - 6 \cdot 3^{n-2} + 6 \cdot 2^{n-2}.$$

Factor powers:

$$= 3^{n-2}(15 - 6) - 2^{n-2}(10 - 6) = 9 \cdot 3^{n-2} - 4 \cdot 2^{n-2}.$$
$$= 3^n - 2^n.$$

Thus the formula holds for $n + 1$.
**Conclusion.** By induction, the closed form $c_n = 3^{n-1} - 2^{n-1}$ is valid for all $n \geqslant 1$. ☺

**Example 3.2.4** (Binary Strings of length m)

How many binary strings of length $n$ has $k$ 1s?

$$B_{k,m} = \text{ number of binary strings with } k \text{ 1s}$$

**Example 3.2.5**

Consider path that go north by length 1 or go northeast with length $\sqrt{2}$. How many ways can you go from $(0, 0)$ to $(n, k), n, k \in \mathbb{N}$?

$$W_{k,n} = \text{ number of such path from } (0, 0) \text{ to } (k, n)$$

$$W_{k,n} = W_{k,n-1} + W_{k-1,n-1} \quad W_{0,n} = 1, W_{k,0} = 0 (k > 0)$$

**Example 3.2.6**

Let $A_{k,n}$ be the number (equivalently, the coefficient) of the term $x^k y^{n-k}$ in the expansion of $(x + y)^n$.
**Claim.** $A_{k,n} = \binom{n}{k}$ for $0 \leq k \leq n$ (and $A_{k,n} = 0$ otherwise).
**Reason 1 (Combinatorial).** In $(x + y)^n$, to get $x^k y^{n-k}$ you must choose exactly $k$ of the $n$ factors to contribute an $x$ (the others contribute $y$). There are $\binom{n}{k}$ such choices.
**Reason 2 (Pascal recurrence).** Define $A_{k,1}$ from $(x + y)^1 = x + y$: $A_{0,1} = 1$, $A_{1,1} = 1$. Multiply by $(x + y)$ to go from $n$ to $n + 1$:

$$(x + y)^{n+1} = (x + y)(x + y)^n.$$

The coefficient of $x^k y^{(n+1)-k}$ in $(x + y)^{n+1}$ comes from: - taking $x$ and the term $x^{k-1} y^{n-(k-1)}$ from $(x + y)^n$ (count $A_{k-1,n}$ ways), - taking $y$ and the term $x^k y^{n-k}$ from $(x + y)^n$ (count $A_{k,n}$ ways).
Thus

$$A_{k,n+1} = A_{k-1,n} + A_{k,n} \quad \text{for } 1 \leq k \leq n,$$

with edge conditions

$$A_{0,n} = 1, \qquad A_{n,n} = 1.$$

This is Pascal's triangle, so $A_{k,n} = \binom{n}{k}$.

**Conclusion.** The number of terms of the form $x^k y^{n-k}$ in $(x+y)^n$ (i.e., its coefficient) is

$$\boxed{A_{k,n} = \binom{n}{k}}.$$

**Example 3.2.7**

How many subsets of size $k$ does $\{1, 2, \ldots, n\}$ have?
**Solution.** A subset of size $k$ is obtained by choosing exactly $k$ distinct elements from the $n$-element set. The number of ways to do this is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

**Reasoning.** Each subset of size $k$ corresponds to a $k$-element combination out of $n$. Order does not matter, so we divide out the $k!$ permutations of the chosen elements.
**Conclusion.** The set $\{1, 2, \ldots, n\}$ has

$$\boxed{\binom{n}{k}}$$

distinct subsets of size $k$.

**Theorem 3.2.3**

If $S_{k,n}$ is the kth entry in the $(n+1)$st rows of pascal's triangle then $S_{k,n} = \frac{n!}{k!(n-k)!}$. We also write $S_{k,n} = \binom{n}{k}$, and we say "$n$ choose $k$".

***Proof:*** By induction,

$$P(n) = \text{"For all } k \in \{0, \ldots, n\} \text{ the kth entry in the } (n+1)\text{st row of Pascals triangle equals:}$$

$$\frac{n!}{k!(n-k)!}$$

😊

**Note:-**

(P1)-(P12) are satisfied by both $\mathbb{Q}$ and $\mathbb{R}$

**Lemma 3.2.1**

1. $n \in \mathbb{N} \Rightarrow n$ is even or odd
2. $n$ even $\Rightarrow n^2$ even
3. $n$ odd $\Rightarrow n^2$ odd
4. $n^2$ even $\Rightarrow n$ even
5. $n^2$ odd $\Rightarrow n$ odd

**Claim 3.2.2**

$\sqrt{2}$ is not rational.
The proof is in notes for baby rudin.

# Chapter 4

# Week 3

## 4.1 Supremum axiom

**Note:-**
This section covered in baby Rudin notes. The following are course notes

Idea: Given a set that has an ordering, we should be able to talk about upper bounds, lower bounds, least upper bounds, greatest lower bounds.

**Example 4.1.1**

$$A = \{2n : n \in \mathbb{N}\} \subseteq \mathbb{Z} \quad \mathbb{Z} \text{ has ordering}$$

- upperbounds:
- lowerbounds: 2,-5,

**Example 4.1.2** (special case)

$$A = \emptyset \subset \mathbb{Q}$$

- GLB LUB doesn't exist
- upperbound & lowerbound: $\mathbb{Q}$

**Definition 4.1.1: Spivak**

$A \subseteq \mathbb{R}$ is **bounded above** if there is an $x \in \mathbb{R}$ so that $x \geqslant a$ for all $a \in A$ such an $x$ is called an **upper bound of** $A$.

**Definition 4.1.2: Spivak**

$A \subseteq \mathbb{R}, x \in \mathbb{R}$ is a **least upper bound of** $A$ if

- $x$ is an upper bound of $A$
- $y$ is an upper bound of $A$ then $x \leqslant y$

We write $x = \text{lub}A$ or we write $x = \sup A$.

**Theorem 4.1.1**

If $A$ has a least upper bound, then it's unique.

***Proof:*** Assume $x$ and $y$ are both $LUB$.

- $x \leqslant y$ because $y$ is an upper bound and $x$ is a least upper bound.
- $y \leqslant x$ because $x$ is an upper bound and $y$ is a least upper bound.

Therefore $x \leqslant y$ and $y \leqslant x$, so $x = y$.

**Note:-**

For quiz, do this with properties (P10-12)

😊

**Definition 4.1.3: (P13) Least upper bound property**

For $A \subseteq \mathbb{R}$ and $A$ **is not empty**. If $A$ is bounded above then $A$ has least upper bound. $\mathbb{R}$ with usual ordering satisfies $(P13)$.

**Definition 4.1.4**

For a ordered field, $a$ is positive if $a \in P$. $a$ is negative if $-a \in P$. Else, $a$ is 0.

**Claim 4.1.1** Natural numbers are not bounded above $\mathbb{R}$

***Proof:*** Suppose, for contradiction, that $\mathbb{N}$ is bounded above in $\mathbb{R}$. Let $s = \sup \mathbb{N}$ (least upper bound), which exists by completeness of $\mathbb{R}$.
Since $s - 1$ is not an upper bound, there exists $m \in \mathbb{N}$ with

$$m > s - 1.$$

But then

$$m + 1 \in \mathbb{N} \qquad \text{and} \qquad m + 1 > s,$$

which contradicts that $s$ is an upper bound for $\mathbb{N}$.
Hence $\mathbb{N}$ has no upper bound in $\mathbb{R}$. 😊